A Framework for Control System Design Subject to Average Data-rate Limits

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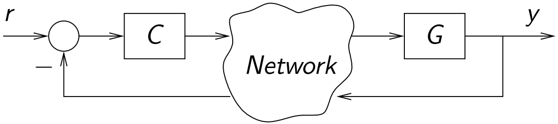
Networked Control Systems (NCSs)



- NCSs are control systems with communication constraints.
- Key questions in NCSs relate to how channel/network artifacts affect control objectives.
- Typical artifacts include data-rate limits (quantization), random time delays and data dropouts.

Background: Can information theory help us?

- "Straight out of the box" information theory does not help: Most information theoretic results do not take causality, stability or delays into account, and are asymptotic in nature.
- Nonetheless, there has been advances in NCS subject to average data-rate constraints.

Background: stability

- Key result (Data-Rate Theorem; Nair and Evans, 2004): In a stochastic setting, and if $\mathcal{R}$ is the average data-rate at which data is transmitted over the feedback path,

$$\mathcal{R} > \sum_{i=1}^{n_p} \ln |p_i|$$

is necessary and sufficient for Mean Square Stability (MSS).

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is necessary and sufficient for Mean Square Stability (MSS).

- The coding schemes that achieve average data-rates close to the above minimum are complex.

Background: Performance guarantees

- Key result (Tatikonda et al, 2004; Nair et al, 2007): For fully observed plants, and quadratic criterion, certainty equivalence and (quasi-)separation holds.
- However, no explicit characterization is provided in Nair et al, 2007.
- For fully observed Gaussian first order autoregressive systems, Tatikonda et al, 2004, provides an expression for a causal rate-distortion function. It is not clear if the results are operationally tight or not.

This Talk

In this talk we will:

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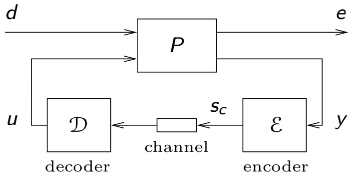
- Present a framework to deal with average data-rate constrained control systems.
- Derive achievable bounds on the average data rate needed to achieve a certain performance level for a specific NCS.

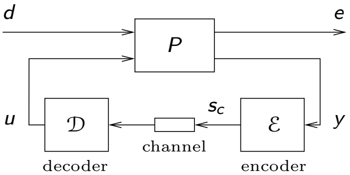
Outline

1. Problem Setup
2. A lower bound on average data-rates
3. Independent source coding schemes
4. Interplay between average data-rates and MSS/performance
5. Conclusions

Problem Setup: Assumptions



- P is LTI and has been designed assuming $u = y$.
- Open loop transfer from u to y is SISO and strictly proper.
- Initial state and disturbance d are jointly Gaussian, second order; d is stationary with PSD $S_d = \Omega_d \Omega_d^H$.



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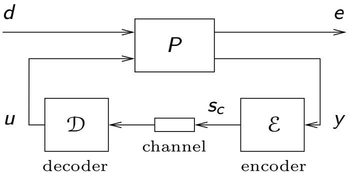
- Open loop transfer from u to y is SISO and strictly proper.
- Initial state and disturbance d are jointly Gaussian, second order; d is stationary with PSD $S_d = \Omega_d \Omega_d^H$.
- The channel is a noiseless digital channel with countable alphabet.
- To be able to use the channel, proper encoding and decoding is needed.

Problem Setup: Our aim



- Study the interplay between the average rate at which s_c is generated and the performance and stability of the NCS.

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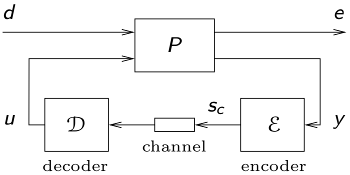
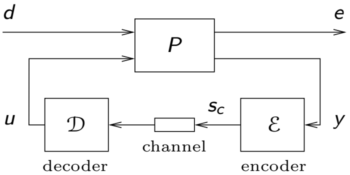


- Study the interplay between the average rate at which s_c is generated and the performance and stability of the NCS.
- By stability we mean MSS, i.e.,

$$\lim_{k \to \infty} \mathcal{E}\left\{ \bar{x}(k)\bar{x}(k)^T \right\} < \infty,$$

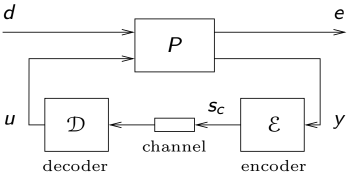
where $\bar{x}$ is the state of the dynamic system in the figure.

Problem Setup: Our aim



- Study the interplay between the average rate at which s_c is generated and the performance and stability of the NCS.
- As performance measure we use the stationary variance of e, namely σ_e^2.

Problem Setup: Our aim

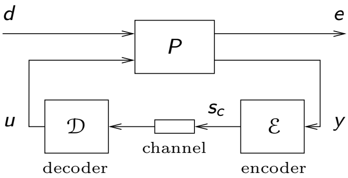


- Study the interplay between the average rate at which s_c is generated and the performance and stability of the NCS.
- Average data rate refers to

$$\mathcal{R} \triangleq \lim_{k\to\infty} \frac{1}{k} \sum_{i=0}^{k-1} R(i),$$

where $R(i)$ is the expected length (in, e.g., bits or nats) of the symbol sent at time instant i, i.e., of $s_c(i)$.

A lower bound on average data-rates



- We focus on causal encoders and decoders:

$$s_c(k) = \mathcal{E}_k(y^k, s_c^{k-1}, S_{\mathcal{E}}^k), \quad u(k) = \mathcal{D}_k(s_c^k, S_{\mathcal{D}}^k),$$

where $S_*(k)$ is side info available at $*$ at time k, and $x^k \triangleq x(0), \cdots, x(k)$.

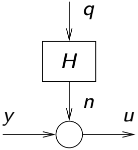
A lower bound on average data-rates

- If $S_{\mathcal{D}} \perp (d, x_o)$ and, upon knowledge of $S_{\mathcal{D}}^k$, $\mathcal{D}_k$ is invertible, then

$$\mathcal{R} \geq I_\infty(y \to u) \triangleq \lim_{k\to\infty} \frac{1}{k} \sum_{i=0}^{k-1} I(u(i); y^i | u^{i-1}).$$

  where $I(a, b|c)$ denotes the conditional mutual information.
- $I_\infty(y \to u)$ is the directed mutual information rate between y and u (Massey, 1990).

Independent source coding schemes: Definition



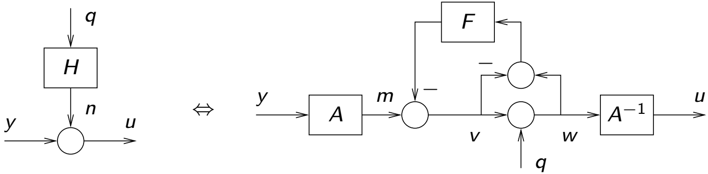
A source coding scheme is independent iff:

- $S_{\mathfrak{D}} \perp (d, x_o)$, $\mathfrak{D}_k$ is invertible given $S_{\mathfrak{D}}^k$, and
- The noise $n \triangleq u - y$ satisfies

$$n = \Omega \cdot q,$$

where q is an i.i.d. zero mean second order sequence, $q \perp (d, x_o)$, and $\Omega \in \mathcal{U}_\infty$ (stable, minimum phase and biproper) has deterministic initial state.

- Equivalent rewriting:



where $A \in \mathcal{U}_\infty$, $F \in \mathcal{RH}_2$ (stable strictly proper).

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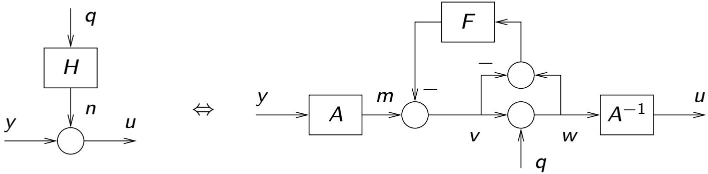


where $A \in \mathcal{U}_\infty$, $F \in \mathcal{RH}_2$ (stable strictly proper).

- In addition, $I_\infty(y \to u) = I_\infty(v \to w)$.

  Characterizing $I_\infty(v \to w)$ is not easy. We next provide bounds

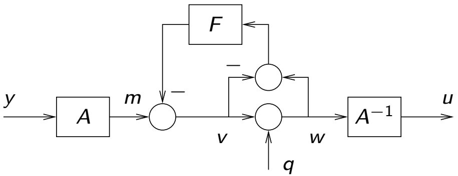
Independent source coding schemes: Bounding $I_\infty(v \to w)$

- In an independent source coding scheme,

$$\frac{1}{4\pi}\int_{-\pi}^{\pi} \ln \frac{S_w}{\sigma_q^2} d\omega \leq I_\infty(v \to w) \leq \frac{1}{4\pi}\int_{-\pi}^{\pi} \ln \frac{S_w}{\sigma_q^2} d\omega + D(q||q_G),$$

  where S_w is the stationary PSD of w and σ_q^2 is the variance of q, and $D(q, q_G)$ is the Kullback-Leibler distance between the distribution of $q(k)$ and its Gaussian counterpart.
- These are simple bounds on the directed mutual information rate across an independent source coding scheme.

Independent source coding schemes: nice, but...



- Q: How to implement the noise q in an independent source coding scheme?

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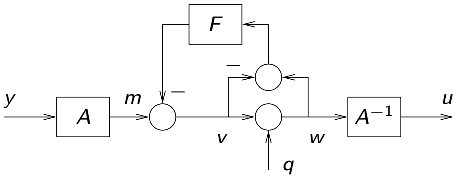
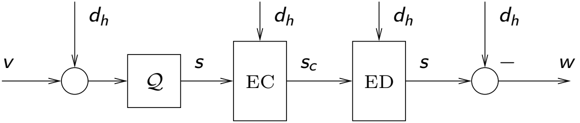


- Q: How to implement the noise q in an independent source coding scheme?
- A: use an entropy coded dithered quantizer (ECDQ; Zamir and Feder, 1992).

Independent source coding schemes: ECDQs

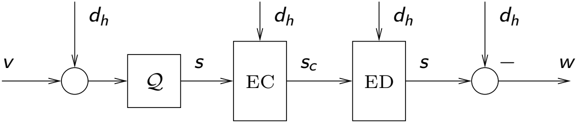


- $\mathcal{Q}$ is a scalar uniform quantizer

$$Q(x) = i\Delta, \quad \Delta\left(i - \frac{1}{2}\right) \leq x < \Delta\left(i + \frac{1}{2}\right), i \in \mathbb{Z}.$$

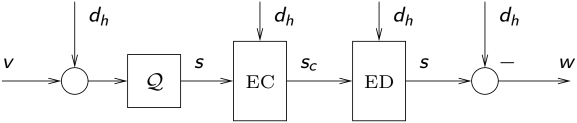
- The dither d_h is i.i.d., $d_h \perp (d, x_o)$, and $d_h(k) \sim \mathcal{U}\left(-\frac{\Delta}{2}, \frac{\Delta}{2}\right)$.

Independent source coding schemes: ECDQs



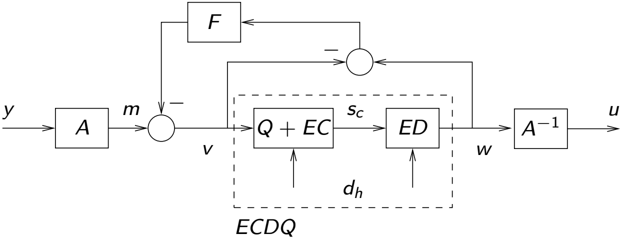
- EC and ED constitute a loss-less entropy coding pair that work conditioned upon (d_h^k, s^{k-1}).
- Since EC and ED are loss-less, the "outside world" is not affected by their choice.

Independent source coding schemes: ECDQs



- For any EC and ED, even if there exits strictly causal feedback from w to v (extension of Zamir and Feder 1992, 1995):
  - $q \triangleq w - v$ is i.i.d.,independent of (d, x_o), and second order provided $\Delta < \infty$.
  - $q(k) \sim \mathcal{U}\left(-\frac{\Delta}{2}, \frac{\Delta}{2}\right)$. Thus, $\sigma_q^2 = \Delta^2/12$.

Independent source coding schemes: ach. avg. data-rates

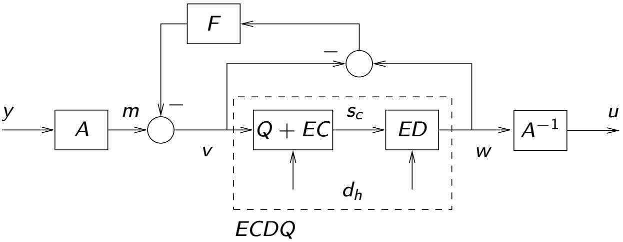


- It can be shown that

$$I_\infty(v \to w) \leq \mathcal{R} < I_\infty(v \to w) + \ln 2,$$

where the gap originates in the inefficiency of the EC (extension of Zamir et al, 2008).

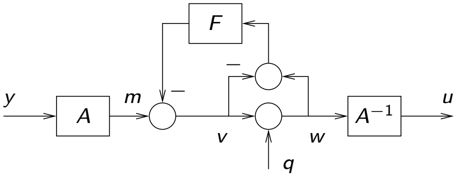
Independent source coding schemes: ach. avg. data-rates



- Thus,

$$\frac{1}{4\pi}\int_{-\pi}^{\pi} \ln \frac{S_w}{\sigma_q^2} d\omega \le \mathcal{R} < \frac{1}{4\pi}\int_{-\pi}^{\pi} \ln \frac{S_w}{\sigma_q^2} d\omega + \frac{1}{2}\ln\left(\frac{2\pi e}{12}\right) + \ln 2,$$

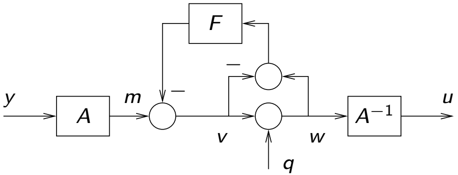
where the second term is the divergence of a uniform distribution from Gaussianity, and the third one is the inefficiency of the EC.



Since an independent source coding scheme has sufficient degrees of freedom,

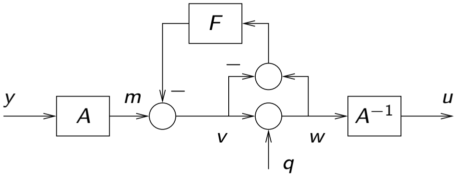
- One can always assume, w.l.o.g., that w is white, which also implies that
- the EC-ED pair need only work conditioned upon $d_h(k)$ (memoryless ECDQ).

Independent source coding schemes: recap



- For any independent source coding scheme that uses an ECDQ,

$$\frac{1}{4\pi}\int_{-\pi}^{\pi}\ln\frac{S_w}{\sigma_q^2}d\omega \leq \mathcal{R} < \frac{1}{4\pi}\int_{-\pi}^{\pi}\ln\frac{S_w}{\sigma_q^2}d\omega + \frac{1}{2}\ln\left(\frac{2\pi e}{12}\right) + \ln 2,$$



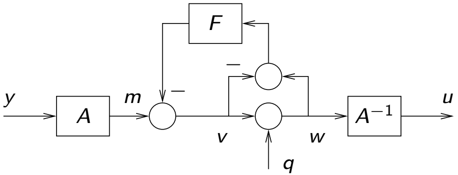
- For any independent source coding scheme that uses an ECDQ,

$$\frac{1}{2}\ln(1+\gamma) \leq \mathcal{R} < \frac{1}{2}\ln(1+\gamma) + \frac{1}{2}\ln\left(\frac{2\pi e}{12}\right) + \ln 2,$$

where

$$\gamma \triangleq \frac{\sigma_v^2}{\sigma_q^2}$$

is the SNR of the independent source coding scheme.



- For any independent source coding scheme that uses an ECDQ,

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- Thus, if we study the interplay between γ and performance (or MSS), then we will be able to say something about the interplay between $\mathcal{R}$ and performance (or MSS).

Average data-rates and MSS:

- Minimal SNR compatible with MSS (Braslavsky et al, 2007; Silva et al, 2010):

$$\inf_{\substack{A,F,\sigma_q^2\\ MSS}} \gamma = \prod_{i=1}^{n_p} |p_i|^2 - 1.$$

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- Thus, the minimal avg. data-rate compatible with stability satisfies

$$\sum_{i=1}^{n_p} \ln |p_i| \leq \inf_{\substack{A,F,\Delta \\ EC-ED \\ MSS}} \mathcal{R} < \sum_{i=1}^{n_p} \ln |p_i| + \underbrace{\frac{1}{2} \ln \left(\frac{2\pi e}{12} \right) + \ln 2}_{1.254 \text{bits}} .$$

Just 1.254 bits/sample away from absolute minimal average data-rate for MSS.

Average data-rates and performance: Prelude I

We first consider the problem of finding, for $D \in \mathbb{R}^+$,

$$\gamma_D \triangleq \inf_{\substack{A, F, \sigma_q^2 \\ MSS,\ \sigma_e^2 \leq D}} \gamma.$$

Average data-rates and performance: Prelude II

If $||T_{de}\Omega_d||_2^2 < D < \infty$, then

$$\gamma_D = h(\lambda_D) \triangleq \exp\left(\frac{1}{\pi}\int_{-\pi}^{\pi} \ln\left(\sqrt{\frac{Y^2}{\lambda_D} + |S|^2} + \frac{Y}{\sqrt{\lambda_D}}\right) d\omega\right) - 1,$$

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where $\lambda_D \in \mathbb{R}^+$ is such that

$$g(\lambda_D) \triangleq ||T_{de}\Omega_d||_2^2 + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\lambda_D Y}{2\left(\sqrt{Y^2 + \lambda_D |S|^2} + Y\right)} d\omega = D,$$

and Y, S, T_{de} are given functions of P.

(Finding λ_D is easy because g is convex and monotone)

Average data-rates and performance: main result

- Now, we will focus on average data-rates:

$$\mathcal{R}_D \triangleq \inf_{\substack{A,F,\Delta \\ EC-ED \\ MSS,\ \sigma_e^2 \leq D}} \mathcal{R}.$$

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- If $||T_{de}\Omega_d||_2^2 < D < \infty$, then

$$\frac{1}{2}\ln(1+\gamma_D) \leq \mathcal{R}_D < \frac{1}{2}\ln(1+\gamma_D) + \frac{1}{2}\ln\left(\frac{2\pi e}{12}\right) + \ln 2,$$

where γ_D is as before.

Average data-rates and performance: remarks

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Average data-rates and performance: remarks

- To the best of our knowledge, our results are the first results that, in closed form, provide insights into the interplay between achievable average data-rates and performance in NCS.
- If (x_o, d) is not Gaussian, then the upper bounds on $\mathcal{R}$ still hold, but it is difficult to characterize lower bounds.
- Warning: our bounds may be very conservative when compared with the, still unknown, minimal values of $\mathcal{R}$ that are achievable when causal but otherwise unconstrained source coding schemes are employed.

Conclusions

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Conclusions

- We established a (restricted) bridge between control and information theories using the notion of independent source coding scheme.
- We used that bridge to analyze the interplay between average data-rates and MSS and performance of a specific NCS.
- A key aspect of our approach is that the internal coding scheme SNR plays a fundamental role.
- Open problems:
  - Consider causal but otherwise unconstrained source coding schemes.
  - Consider general control architectures (partly solved).

Related papers

- E.I. Silva, M.S. Derpich, J. Ostergaard. *A framework for control system design subject to average data-rate constraints*. Conditionally accepted in IEEE Transactions on Automatic Control, March 2010.
- E.I. Silva, M.S. Derpich, J. Ostergaard. *Best LTI Performance under Average Data-rate Constraints*. Submitted to the IFAC Workshop on Distributed Estimation and Control in Networked Systems, 2010.
- E.I. Silva, M.S. Derpich, J. Ostergaard. *An achievable data-rate region subject to a stationary performance constraint for LTI plants*. To be submitted.

Thanks for your attention.