Improved Upper Bounds to the Causal Quadratic Rate-Distortion Function for Gaussian Stationary Sources

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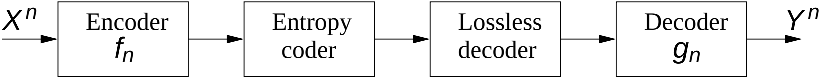
Conventional Non-Causal Rate-Distortion Theory

The Setting



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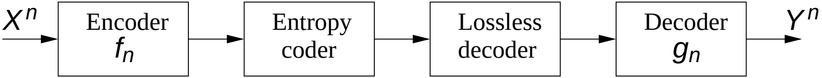

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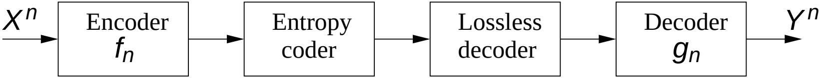

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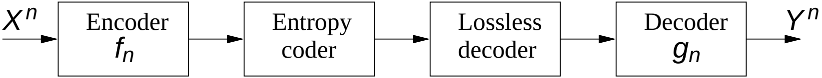

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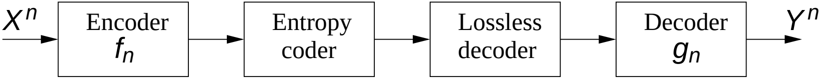

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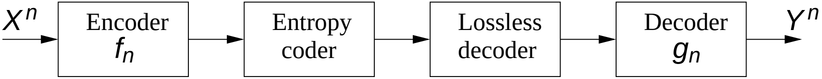

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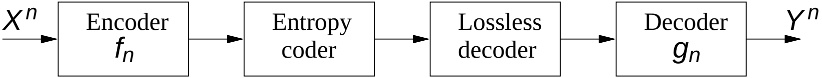

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- Expected Distortion: $d(X^n, Y^n)$
- For example: $d(X^n, Y^n) = \mathbb{E}[\frac{1}{n}\|X^n - Y^n\|^2]$ (MSE)

Rate-Distortion Function

- The Rate-Distortion Function (RDF) $R(D)$ has an operational meaning:
  1. Form all encoder-decoder pairs $\{(f_n, g_n)\}$ that achieve $d(X, g_n(f_n(X^n))) \leq D$
  2. The pair(s) that yields the minimum rate (after entropy coding) define $R(D)$

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- The information-theoretic RDF $R^{it}(D)$ for the source X^n under $d(X^n, Y^n)$ is defined as:

$$R^{it}(D) = \lim_{n \to \infty} \frac{1}{n} \inf I(X^n; Y^n)$$

where the infimum is over all conditional distributions $P(Y^n | X^n)$ such that $d(X^n, Y^n) \leq D$

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- No minimization over encoders/decoders is necessary!

Information-Theoretic RDF

- The RDF for an i.i.d. source X^n for bounded $d(X^n, Y^n)$ is **equal** to the *information-theoretic* RDF [Cover & Thomas]

$$R(D) = R^{it}(D)$$

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- This holds generally for stationary sources and *nice* distortion measures
- A key aspect that allows one to achieve $R^{it}(D)$ is the freedom to work on infinite-dimensional source vectors and thereby exploit e.g., AEP, instead of having to construct *all* practical codes

Introduction to Causal Source Coding

Definition of Causal Source Codes

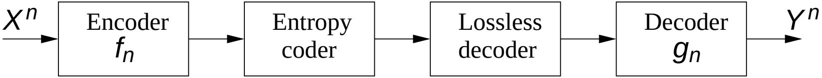
[Neuhoff & Gilbert '82]

- A pair (f_n, g_n) is said to be causal iff

$$g_n(f_n(x^\infty)) = g_n(f_n(\tilde{x}^\infty)), \quad \text{whenever} \quad x^n = \tilde{x}^n, \forall n \in \mathbb{Z}^+$$

- Equivalently, iff the following Markov chain holds:

$$X^\infty \leftrightarrow X^n \leftrightarrow Y^n, \quad \forall n \in \mathbb{Z}^+$$



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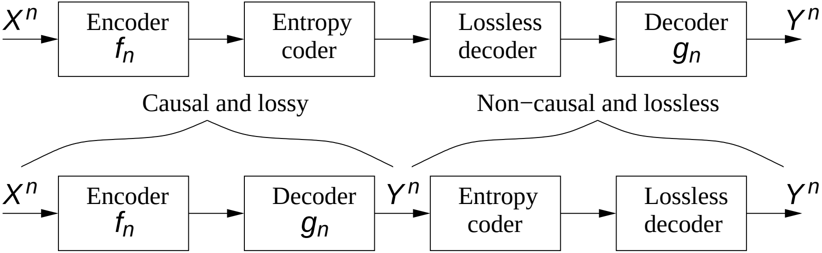

Operational Causal Rate Distortion Function

- Average operational rate of the causal encoder-decoder:

$$r(X^\infty, Y^\infty) \triangleq \limsup_{n\to\infty} \frac{1}{n}\mathbb{E}[L_n(X^\infty)]$$

where $L_n(X^\infty)$ is the total number of bits the decoder has received when reconstructing Y^n

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- Operational *Causal* RDF

$$R_c^{op}(D) \triangleq \inf r(X^\infty, Y^\infty), \quad \text{such that } d(X^\infty, Y^\infty) \leq D$$

where the infimum is over all causal coders (f_n, g_n), and where the entropy-coder may be non-causal

Information-Theoretic Causal Rate Distortion Function

- Information-theoretic *causal* RDF

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- Special case of Pinsker's and Gorbunov's *non-anticipative epsilon-entropy* '87
- Converges to Shannon's QG RDF as $R \to \infty$ [Pinsker & Gorbunov '87,'91]

Existing Results

$$R_c^{op}(D) \overset{(a)}{\geq} R_c^{it}(D) \overset{(b)}{\geq} R(D)$$

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- Generally not equality in $(a) \Rightarrow$ the operational causal RDF is not always equal to the information-theoretic causal RDF

A Recent Closed-Form Result

Sequential RDF for first-order Gauss-Markov source [Tatikonda, Sahai, Mitter, ’00, ’04]:

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  where $\{\xi_n\}$ i.i.d. zero-mean Gaussian with variance σ_ξ^2
- In this case

$$R^{it}_{SRDF}(D) = \max\left\{0, \frac{1}{2}\log_2\left(a^2 + \frac{\sigma_\xi^2}{D}\right)\right\} \text{ bits/sample}$$

Bounding The Rate Loss Due To Causality

Theorem 1

For a 1st-order Gauss-Markov source

$$X_{n+1} = aX_n + \xi_n,$$

under an average MSE distortion constraint, we have

$$\begin{aligned} R_c^{it}(D) &= R_{SRDF}^{it}(D) \\ &= \max\left\{0, \frac{1}{2}\log_2\left(a^2 + \frac{\sigma_\xi^2}{D}\right)\right\} \textit{bits/sample} \end{aligned}$$

for any $D > 0$ and $|a| < 1$. $R_c^{it}(D)$ is realized by a Gaussian error process $\{Z_n\}$, which is *jointly* stationary with $\{X_n\}$.

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The fifth (and tightest) bound must be found by iteration.

Another Causal Information Theoretic RDF $\bar{R}_c^{it}(D)$

Definition

$$\bar{R}_c^{it}(D) \triangleq \inf \bar{I}(X^\infty; Y^\infty)$$

where the infimum is over all processes Y^∞ such that:

i) $d(X^\infty, Y^\infty) \leq D$,

ii) $X^\infty \leftrightarrow X^n \leftrightarrow Y^n$, $\forall n \in \mathbb{Z}^+$ (causality)

iii) the reconstruction error $Z^\infty \triangleq Y^\infty - X^\infty$ is jointly stationary with the source

Clearly:

- $R_c^{it}(D) \leq \bar{R}_c^{it}(D)$
- For first-order Gauss-Markov sources: $R_c^{it}(D) = \bar{R}_c^{it}(D)$

Conjecture: For *m*th-order Gauss-Markov sources, $R_c^{it}(D) = \bar{R}_c^{it}(D)$ (this is an open problem!)

The Bound $B_1(D)$

Theorem 2 For the mth-order Gauss-Markov source and positive D,

$$\bar{R}_c^{it}(D) - R(D) \leq B_1(D) \triangleq \frac{1}{2}\log_2\left(\left[\sum_{i=1}^{m}|a_i|\right]^2 + \frac{\sigma_\xi^2}{D}\right) - R(D),$$

with equality for $m = 1$.

Theorem 3 (bounds B_2–B_4)

Let $\{X_n\}$ be a stationary zero-mean Gaussian source with PSD $S_X(e^{j\omega})$, variance σ_X^2 and such that

$$\eta_X^2 \triangleq \frac{1}{2\pi}\int_{-\pi}^{\pi} S_X(e^{j\omega})^{-1} d\omega$$

exists (is finite and positive). Then...

Theorem 3 (bounds B_2–B_4)

...

$$R_c^{it}(D) - R(D) \leq \bar{R}_c^{it}(D) - R(D) \leq B_2(D) \leq B_3(D) < B_4(D)$$

where all rates are in [bits/sample], and where

$$B_2(D) \triangleq R^{\perp}\left(\frac{\sigma_X^2 D}{\sigma_X^2 - D}\right) - R(D)$$

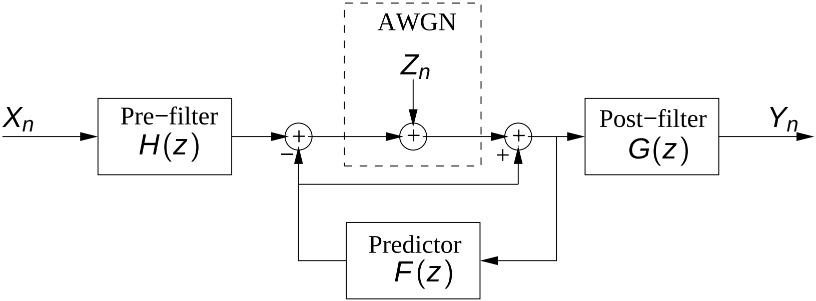
$$B_3(D) \triangleq \frac{1}{4\pi}\int_{-\pi}^{\pi} \log_2\left(1 + \left[1 - \frac{D}{\sigma_X^2}\right]\frac{S_X(e^{j\omega})}{D}\right) d\omega - R(D)$$

$$B_4(D) \triangleq \begin{cases} \frac{1}{2}\log_2(1 + D\eta_X^2), & \text{if } D \leq \frac{1}{\eta_X^2} \\ 0.5, & \text{if } \frac{1}{\eta_X^2} < D < \frac{\sigma_X^2}{2} \\ \frac{1}{2}\log_2(\frac{\sigma_X^2}{D}), & \text{if } \frac{\sigma_X^2}{2} \leq D \leq \sigma_X^2 \end{cases}$$

Obtaining the Information-Theoretic Causal RDF

Achieving the Quadratic Gaussian RDF by Prediction

- Shannon's RDF can be realized by an AWGN channel surrounded by LTI pre-, post- and feedback-filters [Zamir, Kochman, Erez '08]
- The pre- and post-filters are *matched*: $H(z) = G^*(z)$, so at least one of them must be non-causal!



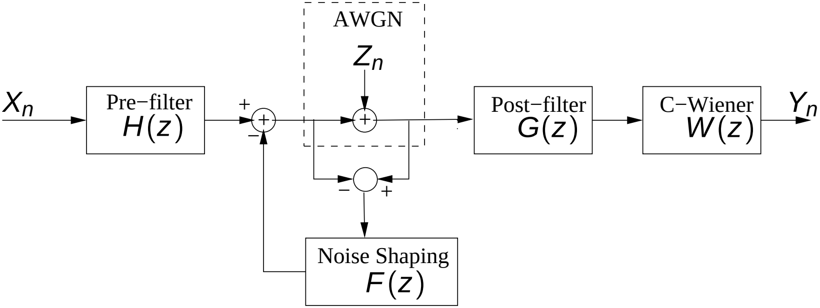
Equivalent test-channel representation

Achieving the Quadratic Gaussian Causal RDF by Prediction

- The optimal filters are known in closed-form (obtained from the forward additive noise channel realization of the RDF) [Zamir, Kochman, Erez '08]
- Assume we make the pre-filter causal – can we then simply replace the post-filter by its causal version and be optimal for the causal regime?
- No — generally not!
- Moreover, the causal Wiener filter is generally not known in closed-form nor is its performance
- What about joint optimization of all the causal filters?
- It is hard to show that the optimization problem is convex in the filter responses of all filters.
- Computationally demanding to optimize over filters having a large number of taps

Achieving the Quadratic Gaussian Causal RDF

- They key idea is to constraint the pre- and post-filters to be inverses of each other: $H(z)G(z) \equiv 1$
- And then add an additional post-filter (MMSE causal Wiener filter)



Achieving the Quadratic Gaussian Causal RDF

Theorem 3

- If the filters that minimize the MSE yield distortion D, subject to an SNR constraint $\gamma \leq \Gamma$ in the inner AWGN channel (Optimization Problem 1), then:

$$\bar{R}_c^{it}(D) = \frac{1}{2}\log_2(\Gamma), \quad \forall D \geq 0$$

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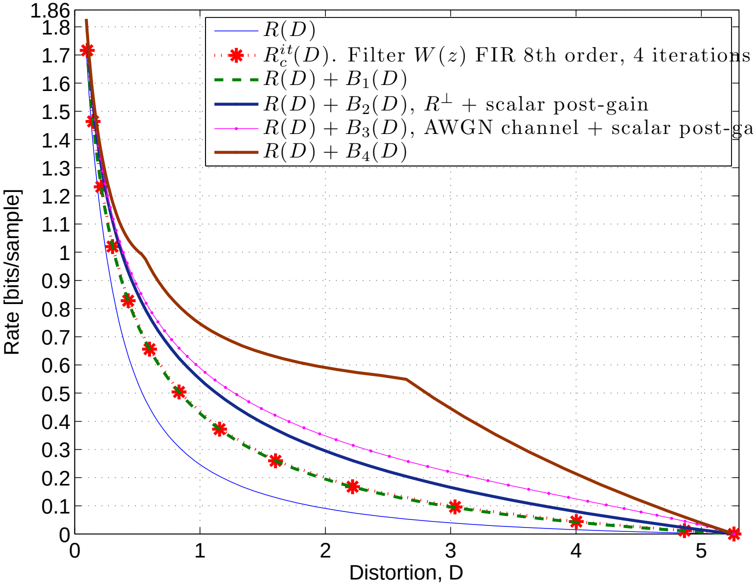
We propose an iterative algorithm which, thanks to these theorems, is guaranteed to converge to $\bar{R}_c^{it}(D)$.

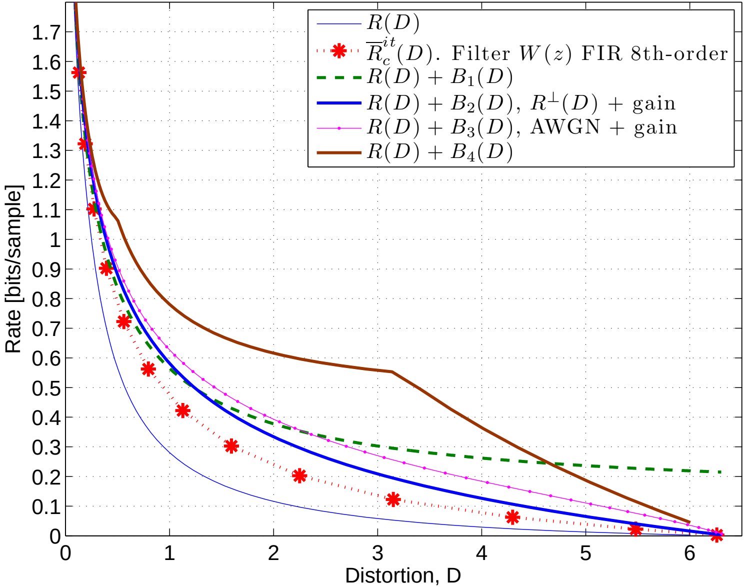
Example 1: First-Order Gauss-Markov Process

- $X_n = 0.9X_{n-1} + \xi_n$



Example 2: Second-Order Gauss-Markov Process:

- $X_n = X_{n-1} - 0.09X_{n-2} + \xi_n$



Observations

- For source 1 (AR-1), $B_1(D)$ is everywhere tight
- For source 2 (AR-2), $B_1(D)$ is generally loose
- For both sources, the maximum gap is:

$$\bar{R}_c^{it}(D) - R(D) < 0.22 \quad \text{bits/dim.}$$

- $B_2(D), B_3(D), B_4(D)$ tend to $\bar{R}_c^{it}(D)$ at low and high rates
- After five iterations for Source 1, the resulting filter taps are:

$$W(z) = 0.3027 + 0.1899z^{-1} + 0.1192z^{-2} + 0.0748z^{-3} + 0.0470z^{-4} + 0.0296z^{-5} + \cdots + 0.0070z^{-9}$$

  Thus, really no need for higher filter orders for this source.
- With a target rate of $R = 0.2601$ bits/sample, the resulting distortions after each iteration were:

$$1.6565, 1.6026, 1.6023, 1.6023,$$

  which suggests that (in this case) the procedure converges (to within a sensible accuracy) after just 3 iterations

Upper Bounds to Operational Causal RDF

- If we replace the AWGN channel by a subtractively dithered entropy-coded uniform (scalar) quantizer we can upper bound the operational causal RDF by:

$$R_c^{op}(D) \leq \bar{R}_c^{it}(D) + 0.254 \quad \text{bits/sample}$$

- where the 0.254 is the space-filling loss of a scalar quantizer

- If we do not allow entropy coding with memory: (Zero-delay — Causal entropy coding) we get

$$R_{ZD}^{op}(D) \leq \bar{R}_c^{it}(D) + 0.254 + 1 \quad \text{bits/sample}$$

Conclusions

- Obtaining the information-theoretic causal RDF (for jointly stationary distortion) is equivalent to optimizing an LTI feedback system for SNR
- This forms a convex optimization problem and we provided a simple iterative algorithm with guaranteed convergence. Thus, no need to compute complicated expressions involving mutual information rates.
- We provided several upper bounds on the difference $R_c^{it}(D) - R(D)$; three of them are always strictly smaller than 0.5 bits/sample for any stationary Gaussian source
- The looser the bound, the easier it is to compute
- Operational upper bound: $R_c^{op}(D) \leq \bar{R}_c^{it}(D) + 0.254$
- Operational upper bound: $R_{ZD}^{op}(D) \leq \bar{R}_c^{it}(D) + 0.254 + 1$